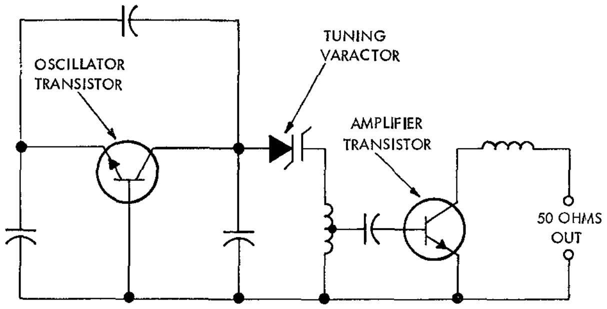

Fig. 14. Oscillator amplifier combination (excluding bias circuitry).

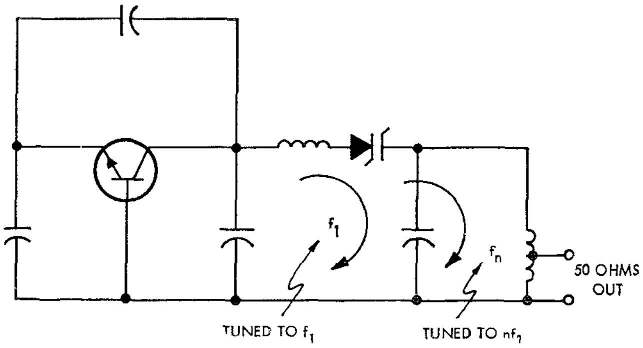

Fig. 15. Transistor oscillator doublers combination (excluding bias circuitry).

V. Conclusions

A common base varactor tuned transistor amplifier circuit has been analyzed and the circuit conditions for proper oscillation have been defined. Two such oscillators were constructed, one at L-band and the other at UHF. Both performed as expected, the results conforming quite well to what the analysis predicted. It is expected that the use of multiple diodes can improve the performance of the circuit. Collector base multiplication can be used to extend this operation into S-band.

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Propagation in a Microwave Model Waveguide of Variable Surface Impedance—Theory and Experiment

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***Abstract*—In this paper propagation in a model terrestrial waveguide is investigated. The surface impedance of the waveguide boundary is assumed to vary along the path of propagation. A quasi-optical approach is used to derive the solution for the case of an abrupt variation in the surface impedance. The reciprocity theorem is employed to facilitate that solution for both directions of propagation. Experimental verification of this technique is obtained from measurements in the model waveguide.**

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I. Introduction

In EARLIER model studies[1,2,3] of radio propagation in a nonuniform terrestrial waveguide, the variations of the electrical properties of the ionosphere

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[3] ——, "Propagation in a model terrestrial waveguide of nonuniform height: theory and experiment," *J. Res. NBS* (*Radio Sci.*), vol. 69D, pp. 1445–1463, November 1965.

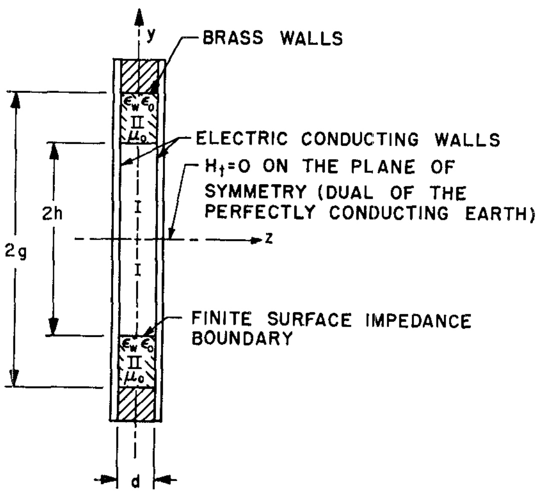

Fig. 1. Cross section of the dual model waveguide with finite surface impedance boundaries.

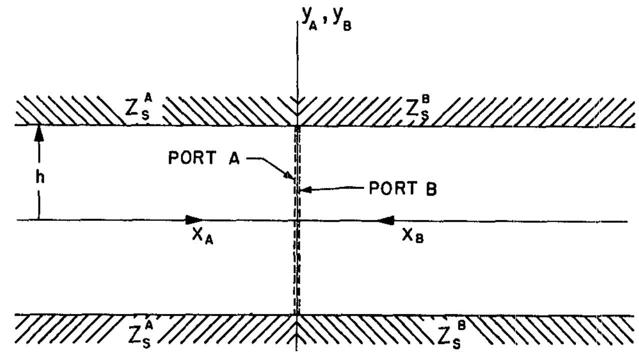

Fig. 2. Waveguide with an abrupt variation in the surface impedance boundaries analyzed as a multiport waveguide junction.

were represented by corresponding variations in the effective height of reflection of the ionosphere boundary along the path of propagation. The ionosphere, considered sharply bounded, was characterized by a constant finite surface impedance boundary.[4,5] While this representation of the nonuniform ionosphere serves as a convenient model to study mode conversion phenomena, it is evident from comparison of the attenuation rates along various paths of propagation at different times of the day[6] that a constant surface impedance boundary cannot, in general, be a good representation of the ionosphere boundary. For the purpose of investigating radio propagation over large distances, it would therefore be more appropriate to characterize the ionosphere by a boundary of variable height and surface impedance.

In this paper the effects of an abrupt variation in the surface impedance of the boundary are investigated in particular, and the height of the ionosphere is assumed constant. Furthermore, as in the two-dimensional dual model discussed earlier,[1,2,4] a flat earth approximation will be assumed and the effects of the earth's magnetic field will be neglected. The effects of a gradual variation in the surface impedance along the path of propagation may also be derived from these results.[7]

The basic experimental tool in this work is the rectangular dual model[8] waveguide whose half height h corresponds to the height of the ionosphere in wavelengths, as shown in Fig. 1. The vertical boundaries (at $z=\pm d/2$) are electric conductors. Only TE_{n0} modes with symmetric electric fields in the (y, z) plane are launched into the system such that the (x, z) plane in the dual model (on which $H_x=0$) corresponds to a perfectly conducting earth. The horizontal finite surface impedance boundaries (at $y=\pm h$) correspond to the surface impedance for grazing modes at the ionosphere boundary.[4] Hence either the upper or lower half of the waveguide corresponds to the earth-ionosphere waveguide.

II. Formulation of the Problem

Consider a multimode waveguide with uniform cross section into which only TE_{n0} modes $(n=1, 3, 5, \cdots)$ are launched, as shown in Fig. 2. The surface impedance Z_S at the narrow boundaries $y=\pm h$ is Z_S^A for $x<0$ and Z_S^B for $x>0$. The transition region ($x=0$ plane) in which the discontinuity of the surface impedance occurs is treated as a two-port waveguide junction.[2] The right-hand coordinate systems connected with Ports A and B are x_a, y_a, z_a, and x_b, y_b, z_b, respectively, such that the y axes coincide with the transition plane and the x axes point toward the waveguide junction.

This class of problems involving waveguide discontinuities can, in general, be reduced to the solution of an infinite set of linear algebraic equations.[9] Solutions of the infinite set of equations have been derived in terms of infinite products. These solutions are based either on a method for inverting the special form of the resulting matrix equation,[10,11] or on the function theoretical method which represents an infinite series of a certain type by a contour integral of the Cauchy type.[12] The quasi-optical solution developed in the paper is expressed in terms of an infinite sum; the higher order terms in the series correspond to correction terms due to finite reflections at the waveguide discontinuity. These

[4] E. Bahar, "Model studies of the influence of ionosphere perturbations on VLF propagation," Dept. of Elec. Engrg., University of Colorado, Boulder, Tech. Rept. ARPA, Contract CST 7348, May 1964.

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[12] H. J. Frankena, "Coupling of two semi-infinite circular waveguides with walls of different surface admittances," presented at the 1965 URSI Symp. on Electromagnetic Wave Theory, Delft, The Netherlands.

solutions have been found to be very suitable in the case of propagation in multimode waveguides in which the surface impedance may vary abruptly from zero to the order of magnitude of the free space wave impedance $\eta=(\mu_0/\epsilon_0)^{1/2}$. The efficiency of this technique is demonstrated by a comparison with experimental results taken from a model waveguide. Furthermore, the quasi-optical solutions may be readily generalized to derive the solution for propagation in waveguide regions in which the surface impedance varies continuously in an arbitrary manner.[7]

This problem is closely related to the extensive work on mixed-path propagation which starts with the compensation theorem and proceeds via a first-order perturbation of an integral equation. The formulation has been applied to the curved earth-ionosphere waveguide.[13] Experimental model studies of propagation of ground waves across mixed paths have also been conducted.[14] A review of the progress in this area (including many references to earlier work) has been presented by Wait.[15]

The modal equation satisfied by a waveguide mode propagating in the region bounded by the surface impedance Z_S^A is[8]

$$\exp\{i2kC_n^Ah\} = \frac{C_n^A - Y_S^A\eta}{C_n^A + Y_S^A\eta} = R_h^A(C_n^A) \tag{1}$$

where k is the free space wavenumber, C_n^A can be identified as the complex angle of incidence of the nth mode on the surface impedance boundaries of the waveguide, Y_S^A is the surface admittance (the reciprocal of the surface impedance Z_S^A), and R_h^A is the reflection coefficient for horizontally polarized waves. Actually Y_S^A is also a function of the mode number, but considering that most of the energy in the multimode waveguide is distributed into the lower order modes, the "constant" surface impedance concept has been employed. For grazing modes, $|C_n^A| \ll |Y_S^A|\eta$ and R_n^A may be approximated by

$$R_h^A \approx -\exp\{-2C_n^AZ_S^A/\eta\}. \tag{2}$$

Hence (1) reduces to

$$C_n^A \approx \frac{n\pi}{2(kh - iZ_S^A/\eta)}, \qquad n = 1, 3, 5 \cdots. \tag{3}$$

Therefore, for modes of grazing incidence in multimode waveguides, $|C_n^A|^2 \ll 1$.

For the cases in which the approximate formula is not appropriate, (1) may be solved by an iterative process that uses (3) as the zero-order approximation.

[13] J. R. Wait, "Influence of an inhomogeneous ground on the propagation of VLF radio waves in the earth-ionosphere waveguide," *J. Res. NBS* (*Radio Sci.*), vol. 69D, pp. 969–976, July 1965.

[14] R. J. King, J. R. Wait, and S. W. Maley, "Experimental and theoretical studies of propagation of ground waves across mixed paths," presented at the 1965 URSI Symp. on Electromagnetic Wave Theory, Delft, The Netherlands.

[15] J. R. Wait, *Advances in Radio Research*, vol. 1, J. A. Saxton, Ed London: Academic, 1964, pp. 157–217.

The z-directed electric field for the forward propagating nth mode in the region $x<0$ is

$$E_n^A = a_n^A \exp\{-i\beta_n^Ax\}\Phi_n^A \tag{4}$$

where a_n^A is the complex wave amplitude and Φ_n^A is the basis field

$$\Phi_n^A = A_n^A \cos k_n^Ay. \tag{5}$$

A_n^A is a normalization constant and k_n^A is related to the propagation constant β_n^A and the characteristic admittance Y_n^A through the equations

$$\beta_n^A = [k^2 - (k_n^A)^2]^{1/2} = k[1 - (C_n^A)^2]^{1/2} = kS_n^A \tag{6a}$$

and

$$Y_n^A = S_n^A/\eta \tag{6b}$$

and S_n^A may be identified as the sine of the angle of incidence for the nth mode. An $\exp(i\omega t)$ time dependance is assumed.

Expressions for the modal equation and the electric field in the region $x>0$ are obtained from (1) and (4), respectively, by substituting the superscript B for A. The completeness of the set of waveguide modes (which is an infinite one since the surface impedance concept is used) has been discussed in earlier work.[4,16] Hence, any arbitrary field in the waveguide can be expressed as a linear combination of the basis set. Throughout this work it will be found useful to use matrix notation; hence the following definitions are made.

Let Φ^A and Φ^B denote the basis field row vectors whose elements are Φ_n^A and Φ_n^B, respectively. At Port A the forward and backward wave amplitudes a_n^A and b_n^A are elements of the wave amplitude column vectors a^A and b^A, respectively. The matrix Y^A is the diagonal characteristic admittance matrix whose element Y_n^A is the nth mode characteristic admittance.

The characteristic impedance matrix Z^A is equal to the inverse of the matrix Y^A. The matrix W^A is a diagonal matrix whose element W_n^A is the power normalization factor defined by

$$W_n^A = Y_n^A \iint [\Phi_n^A]^2 dA$$
$$= Y_n^A(A_n^A)^2[1 + \operatorname{sinc} 2k_n^Ah]A \tag{7}$$

where A is the area of the cross section and $(\sin x)/x$ is defined as $\operatorname{sinc} x$.

The symbol S^{AA} is a square reflection scattering matrix, whose element S_{nm}^{HA} is the complex amplitude of the nth reflected mode when the mth mode of unit amplitude is incident on the junction from Port A. The symbol S^{BA} is a square transmission scattering matrix, whose element S_{nm}^{BA} is the complex amplitude of the

[16] R. L. Gallawa, "Propagation in non-uniform waveguides with impedance walls," Dept. of Elec. Engrg., University of Colorado, Boulder, Tech. Rept. ARPA, Contract CST 7348, May 1964.

nth mode transmitted through the junction from Port A.

Similarly, the superscripts A and B are interchanged when the above quantities are related to Port B.

III. Formal Solution

Consider first the case in which the electromagnetic waves are incident on the junction from Port A. The continuity conditions for the transverse electric and magnetic fields are given, respectively, by

$$\Phi^A[I + S^{AA}]a^A = -\Phi^B S^{BA} a^A \tag{8a}$$

and

$$\Phi^A Y^A[I - S^{AA}]a^A = -\Phi^B Y^B S^{BA} a^A. \tag{8b}$$

Premultiply (8a) by $Y^B\tilde{\Phi}^B$ ($\tilde{\Phi}^B$ is the transpose of Φ^B) and integrate over the area of the cross section to get

$$C^{BA}[I + S^{AA}]a^A = -W^B S^{BA} a^A \tag{9}$$

where the element of the matrix C^{BA} is the coupling coefficient C_{nm}^{BA},

$$\begin{aligned} C_{nm}^{BA} &= Y_n^B \iint \Phi_n^B \Phi_m^A dA \\ &= A_n^B A_m^A[\text{sinc}\,(k_n^B - k_m^A)h \\ &\quad + \text{sinc}\,(k_n^B + k_m^A)h]A \end{aligned} \tag{10}$$

and the orthogonal properties of the waveguide modes are employed.

Similarly, premultiply by $\tilde{\Phi}^A$ to get

$$W^A[I - S^{AA}]a^A = -Z^A C^{AB} Y^B S^{BA} a^A \tag{11}$$

where

$$Z^A C^{AB} = (\widetilde{Z^B C^{BA}}). \tag{12}$$

The solution of (8) and (10) for S^{BA} and S^{AA} yields[2]

$$S^{BA} = -[W^B]^{-1} C^{BA}[I + S^{AA}] \tag{13}$$

and

$$S^{AA} = [I - X^{AA}][I + X^{AA}]^{-1} = \Delta^{AA}[I - \Delta^{AA}]^{-1} \tag{14a}$$

where

$$X^{AA} = I - 2\Delta^{AA} \equiv [W^A]^{-1} Z^A C^{AB} Y^B [W^B]^{-1} C^{BA}. \tag{14b}$$

The power normalization constants may be chosen arbitrarily. For convenience in this case let the power normalization matrices equal the identity matrix I. As a result of this choice, it follows that

$$S^{BA} = -C^{BA}[I + S^{AA}] \tag{15}$$

and

$$\Delta^{AA} = \tfrac{1}{2}[I - Z^A C^{AB} Y^B C^{BA}]. \tag{16}$$

Furthermore, the reciprocity theorem for waveguide junctions[3] reduces to the convenient form

$$S_{nm}^{AB} = S_{mn}^{BA}. \tag{17}$$

It should be noted that the equation for the reflection scattering matrix S^{AA} is analogous to the equation for the reflection coefficient in transmission line theory, with X^{AA} replacing the normalized load admittance.

On premultiplying (8a) and (8b) by $Y^A\tilde{\Phi}^A$ and $\tilde{\Phi}^B$, respectively, before integrating over the area of the cross section, it can be similarly shown that

$$S^{AA} = [W^{AA} - I][W^{AA} + I]^{-1} = -[\nabla^{AA}][I - \nabla^{AA}]^{-1} \tag{18a}$$

where

$$W^{AA} = I - 2\nabla^{AA} = C^{AB} Z^B C^{BA} Y^A = [X^{AA}]^{-1}. \tag{18b}$$

In the above expression for S^{AA}, W^{AA} is analogous to the normalized load impedance. Note that Δ^{AA} and ∇^{AA} reduce to zero matrices in the absence of reflections. Either (14a) or (18a) may be expanded into an infinite series representation for S^{AA},[2]

$$S^{AA} = \sum_{p=1}^{\infty} (\Delta^{AA})^P = -\sum_{p=1}^{\infty} (\nabla^{AA})^P. \tag{19a}$$

Also, on comparing (9) with the equation derived from (8a) on premultiplying by $Y^A\tilde{\Phi}^A$ and integrating, it becomes obvious that the matrices C^{AB} and C^{BA} are the inverses of each other.

IV. Use of Reciprocity Theorem to Facilitate the Solution

In general, the above expressions are valid, provided all the characteristic values of Δ^{AA} or ∇^{AA} are less than unity. Similar expressions may be derived for S^{BB}, the reflection scattering matrix in the opposite direction. However, in view of the reciprocity theorem, in order to completely solve this problem, it is sufficient to derive either S^{AA} or S^{BB} using either one of the above series expansions. Hence, in order to make the most efficient use of this perturbational technique, it is necessary to use only the series expansion that converges most rapidly in the particular situation under consideration. If the reflections are very small for a particular direction of propagation, then the reflection scattering matrix S^{AA} may be approximated by the first term of the series

$$S^{AA} \approx \Delta^{AA} \approx -\nabla^{AA} \tag{19b}$$

in which case there is no significant advantage in using either one of the two expansions for S^{AA}. However, the reflections for the two directions of propagation may differ significantly in amplitude if the change in surface impedance is large. This case is analogous to the scattering due to an abrupt change in the height of the waveguide.[2]

In order to illustrate this aspect of the problem, consider the following extreme case. In the region $x<0$, the tangential electric field vanishes at $y=\pm h$ (electric conducting boundaries), and in the region $x>0$, the tangential magnetic field vanishes at $y=\pm h$. This corresponds to $Z_S^A=0$ and $Y_S^B=0$. In this case the modal equation (1) yields

$$\overset{A}{k}_n h = (2n-1)\pi/2 \quad \text{and} \quad \overset{B}{k}_n h = n\pi \qquad n = 1, 2, 3. \tag{20}$$

The basis fields are

$$\overset{A}{\Phi}_m = \frac{1}{(\overset{A}{Y}_m)^{1/2}} \cos \overset{A}{k}_m y \quad \text{and}$$

$$\overset{B}{\Phi}_m = \frac{1}{(\overset{B}{Y}_m)^{1/2}} \cos \overset{B}{k}_m y \tag{21}$$

and the coupling coefficients C_{nm}^{BA} are given by

$$\overset{BA}{C}_{nm} = \left[\frac{\overset{B}{Y}_n}{\overset{A}{Y}_m}\right]^{1/2} [\text{sine } (m+n-\tfrac{1}{2})\pi + \text{sinc } (m-n-\tfrac{1}{2})\pi]$$

$$= \left[\frac{\overset{B}{Y}_n}{\overset{A}{Y}_m}\right]^{1/2} (-1)^{m-n-1} \frac{2(2m-1)}{(2m-1)^2-(2n)^2}. \tag{22a}$$

Note that with these particular values for k_n^A and k_n^B, the sine functions have extremes, which would mean that the coupling between the modes is maximum as may be expected. In particular, for the case in which the principal mode is incident ($m=1$),

$$\overset{BA}{C}_{n1} = (-1)^n \frac{2}{1-4n^2}\left[\frac{\overset{B}{Y}_n}{Y_1^A}\right]^{1/2}. \tag{22b}$$

Similarly, the coupling coefficient for the opposite direction of propagation is

$$\overset{AB}{C}_{nm} = \left[\frac{\overset{A}{Y}_n}{Y^B_m}\right] C_{mn}{}^{BA}$$

$$= \left[\frac{\overset{A}{Y}_n}{Y^B_m}\right]^{1/2} (-1)^{n-m-1} \frac{2(2n-1)}{(2n-1)^2-(2m)^2} \tag{23a}$$

and in particular for the case in which the principal mode is incident,

$$\overset{AB}{C}_{n1} = (-1)^n \frac{2(2n-1)}{(2n-1)^2-4}\left[\frac{\overset{A}{Y}_n}{Y^B_1}\right]^{1/2}. \tag{23b}$$

On examining (22b) and (23b), it is seen that for the case in which the incident principal mode propagates from the region bounded by the electric conductor to the region bounded by the "magnetic conductor," the coupling coefficients C_{n1}^{BA} decrease approximately as $1/n^2$, but for the opposite direction of propagation, it is seen that the coupling coefficients C_{n1}^{AB} decrease approximately as $1/n$. It follows that on propagating from Port A to Port B, less energy is scattered into higher order modes than for the reverse direction of propagation. As a consequence it will now be shown that reflections at Port A are significantly smaller than those at Port B. First examine the expression for X_{n1}^{AA}.

$$\overset{AA}{X}_{n1} = \overset{A}{Z}_n \sum_p^{\infty} \overset{AB}{C}_{np} \overset{B}{Y}_p \overset{BA}{C}_{p1}$$

$$\approx \overset{A}{Z}_n \overset{B}{Y}_1 \sum_p^{\infty} \overset{AB}{C}_{np} \overset{BA}{C}_{p1}$$

$$= \overset{A}{Z}_n \overset{B}{Y}_1 \delta_{n1}. \tag{24}$$

The major contribution to X_{n1}^{AA} comes from the lower order modes (due to C_{p1}^{BA}). Hence, in the above approximation, Y_p^B is replaced by Y_1^B which is rather constant for the lower order modes since $|C_n^B|^2 \ll 1$, (3). Also it has been noted that $C^{AB}C^{BA}=\mathrm{I}$. Furthermore, since $Z_1^A Y_1^B \approx 1$, it follows that $X_{n1}^{AA} \approx \delta_{n1}$; hence the coefficients $\Delta_{n1}{}^{AA}$ are very small. Similarily, since $C_{1p}{}^{AB}$ decreases as $1/p^2$, it follows that

$$\overset{AA}{X}_{1n} = \overset{A}{Z}_1 \sum \overset{AB}{C}_{1p} \overset{B}{Y}_p \overset{BA}{C}_{pn}$$

$$\approx \overset{A}{Z}_1 \overset{B}{Y}_1 \sum C_{1p} C_{pn} \approx \delta_{1n}. \tag{25}$$

Hence the coefficients $\Delta_{in}{}^{AA}$ are also very small. Therefore, the power series expansion S_{n1}^{AA} may, for all practical purposes, reduce to a very few terms. For the particular example (discussed Section V) in which $Z_S^A=0$ and $Z_S^B=\sqrt{2}\eta$, it has been shown that a very good degree of accuracy is maintained if S_{n1}^{AA} is approximated by the first term of the series expansion (19a).

Now for the opposite direction of propagation, consider the expression for X_{n1}^{BB}

$$\overset{BB}{X}_{n1} = \overset{B}{Z}_n \sum \overset{BA}{C}_{np} \overset{A}{Y}_p \overset{AB}{C}_{p1}. \tag{26}$$

Since C_{p1}^{AB} decreases only as $1/p$, it is necessary to consider several higher order modes for which the characteristic admittances Y_p^B differ appreciably from Y_1^B. Hence, the coefficients X_{n1}^{BB} differ considerably from δ_{n1}, and the reflection coefficients S_{n1}^{BB} cannot be approximated by Δ_{n1}^{BB}. Indeed in the particular example discussed in V, the reflection coefficients S_{n1}^{BB} are significantly greater than S_{n1}^{AA}, but once the scattering coefficients S_{n1}^{AA} are evaluated, the rest of the solution to this problem may be obtained using the reciprocity theorem without introducing any approximations. The transmission scattering matrix S^{BA} can now be evaluated using (16). For the opposite direction of propagation, the transmission scattering matrix S^{AB} is obtained using the reciprocity theorem (17). Finally, the reflection scattering matrix S^{BB} (not derived from the series expansion) is evaluated by employing the continuity condition for the electric field at the transition plane ($x=0$). In terms of the matrices defined above, this condition is given by

$$\Phi^B[I + S^{BB}]a^B = -\Phi^A S^{AB} a^B. \tag{27}$$

On premultiplying (27) by $Y^B\tilde{\Phi}^B$ the following equation for S^{BB} is obtained

$$S^{BB} = -[I + C^{BA}S^{AB}]. \tag{28}$$

V. An Example: Comparison Between the Analytical Solution and the Experimental Data

In order to demonstrate the validity of the analytical solution, the numerical solutions were compared with experimental data obtained from the model waveguide. In order to fully appreciate the efficiency of the analytical solution, a rather large variation in the surface impedance is considered in the particular example investigated. The boundaries $y=\pm h$ at Port A are assumed to be perfectly conducting $Z_S^A=0$ (corresponding to a perfectly reflecting ionosphere $R_i=-1$). At Port B the surface impedance for the grazing modes at the boundaries $y=\pm h$ is $Z_S^B=\sqrt{2}\eta$ (a typical value for quiet daytime conditions).[4]

For the case in which the principal mode is incident at Port A, it is seen that the reflection scattering coefficients are all less than 0.01 and hence indistinguishable from experimental error in the model waveguide (Fig. 3). On reversing the direction of propagation of the incident mode, the reflections are found to be considerable, and the field pattern at the plane of discontinuity is significantly different from that of the incident mode. In order to obtain the analytical solution plotted together with the experimental data (Fig. 4), the reflection scattering matrix S^{AA} is first derived assuming $S^{AA}\approx\Delta^{AA}\approx-\nabla^{AA}$ rather than directly evaluating S^{BB} through the series expansion. The rest of the solution is obtained using reciprocity as indicated in the previous section. The scattering coefficients $S_{n1}{}^{AB}$ are tabulated in Table I.

TABLE I
Scattering Coefficients S_{n1}^{AB}

n	Re (S_{n1}^{AB})	Im (S_{n1}^{AB})
1	−1.0037	0.0373
3	0.0053	−0.0568
5	−0.0027	0.0336
7	0.0020	−0.0262
9	−0.0019	0.0246
11	0.0024	−0.0313
13	−0.0308	−0.0023
15	0.0138	0.0010
17	−0.0090	−0.0007
19	0.0065	0.0005

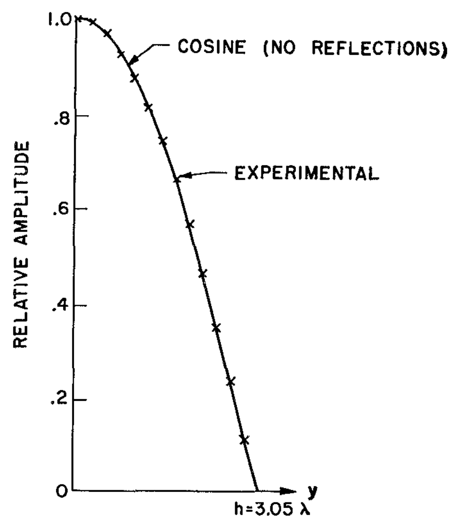

Fig. 3. Electric field pattern at the plane of discontinuity ($x=0$) for the case in which the principal mode is incident from Port A.

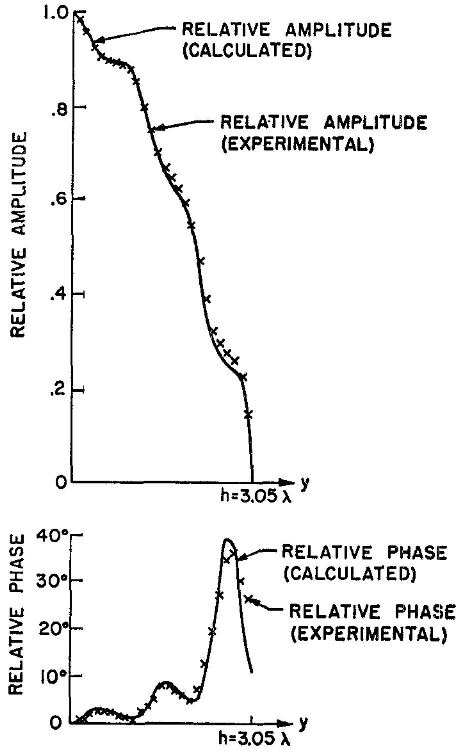

Fig. 4. Electric field pattern at the plane of discontinuity ($x=0$) for the case in which the principal mode is incident from Port B.

VI. Concluding Remarks

It has been shown from the experimental results that a first-order approximation for S^{AA} yields very satisfactory results even for the case of a large variation in the surface impedance. The solution based on the direct evaluation of S^{BB} is far less efficient and leads to poorer results than the solution based on the reciprocity theorem. The quasi-optical approach used in this solution can be further extended to the case in which the surface impedance varies arbitrarily along a finite path.[7] The region in which the surface impedance varies may be considered to consist of an infinity of elementary waveguides. The analysis then leads to an infinite set of coupled differential equations for the forward and backward waves, as in the analysis of waveguides with variable cross sections.[8] It is interesting to note that the technique developed in this paper is not only suitable for large abrupt variations in the surface impedance but also for infinitesimal variations, in which case the exact analytical expression for the reflection scattering matrix S^{AA} reduces from an infinite sum to the first term of the series. This solution may be readily generalized for the case in which different surface impedances characterize the upper and lower boundaries of the waveguide.

Finally, it should be pointed out that nowhere in the derivation of the solution to this problem is it necessary to evaluate directly the inverse of a matrix. Hence, in deriving the numerical solution to a particular problem, it is not a very crucial problem to determine the dimension of the truncated matrices. Since a scattering coefficient of amplitude 0.01 would indicate that the power scattered into the corresponding mode is about 10^{-4} of the total scattered power, modes with smaller amplitudes may be neglected for all practical purposes. It has also been pointed out that power from the incident principal mode will be essentially scattered into the lower order modes; therefore, modes for which $|C_n|^2 \gg 1$ should not be considered, as the numerical example clearly illustrates. This, moreover, justifies the applicability of the "constant" finite surface impedance concept referred to in Section II.

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Correspondence

Multiline 2*N*-Port Directional Couplers

In 1954, Oliver [1] described the basic theory and design of a four-port contradirectional coupler which utilized two sets of coupled transmission lines. We attempt here to generalize this result so as to obtain a $2N$-port contradirectional coupler.

We use the following notation. Capital letters will stand for matrices. The ijth element of a matrix A will be denoted either by a_{ij} or $(A)_{ij}$. The kth element of a column vector $\boldsymbol{a}$ will be denoted by $(\boldsymbol{a})_k$.

Consider a system $N+1$ parallel cylindrical conductors operating in the TEM mode. Since the operation is TEM, we can define a voltage $v_i(x)$ and a current $i_i(x)$ for the ith conductor

$$v_i(x) = -\int_{\text{line } N+1}^{\text{line } i} \boldsymbol{E}\cdot d\boldsymbol{R}\Big|_{x \text{ fixed}} \quad (i = 1, 2, \cdots, N) \tag{1}$$

$$i_i(x) = \oint_{\text{around conductor } i} \boldsymbol{H}\cdot d\boldsymbol{R}\Big|_{x \text{ fixed}} \quad (i = 1, 2, \cdots, N) \tag{2}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ denote the electric and magnetic field vectors. The $N+1$th conductor has been taken as a voltage reference. The situation is shown schematically in Fig. 1.

In practice, the $N+1$th conductor may be a closed waveguide in which the other N conductors are contained, as shown in Fig. 2 in cross section.

Let us define T_z, the transfer impedance matrix of the system, as follows

$$\left[\frac{V_1}{I_1}\right] = T_z\left[\frac{V_2}{I_2}\right] \tag{3}$$

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where

$$V_1 = \begin{bmatrix} v_1(0) \\ v_2(0) \\ \vdots \\ v_N(0) \end{bmatrix} \quad V_2 = \begin{bmatrix} v_1(l) \\ v_2(l) \\ \vdots \\ v_N(l) \end{bmatrix}$$

$$I_1 = \begin{bmatrix} i_1(0) \\ i_2(0) \\ \vdots \\ i_N(0) \end{bmatrix} \quad I_2 = \begin{bmatrix} i_1(l) \\ i_2(l) \\ \vdots \\ i_N(l) \end{bmatrix}.$$

It may be shown [2] that T_z is given by

$$T_z = \frac{1}{\sqrt{1-\lambda^2}}\left[\begin{array}{c|c} 1_N & \lambda W \\ \hline \lambda G & 1_N \end{array}\right], \tag{4}$$

where 1_N is the N by N identity matrix,

$$\lambda = \tanh(j\omega l\sqrt{\mu\epsilon}),$$
$$G = C/\sqrt{\mu\epsilon},$$
$$W = G^{-1} = L/\sqrt{\mu\epsilon},$$

and C and L are the static capacitance and inductance matrices per unit length for the given configuration. L can be determined from C and vice versa [2] since $LC = \mu\epsilon 1_N$. The matrix C is hyperdominant, that is, all its diagonal elements are positive and all its off-diagonal elements are negative [3]:

$$c_{ii} \geq 0. \tag{5}$$

$$c_{ij} \leq 0 \quad i \neq j. \tag{6}$$

Also, all the elements of L are positive:

$$l_{ij} \geq 0. \tag{7}$$

Note that

$$\frac{w_{mn}}{q_{pq}} = \frac{l_{mn}}{c_{pq}}.$$

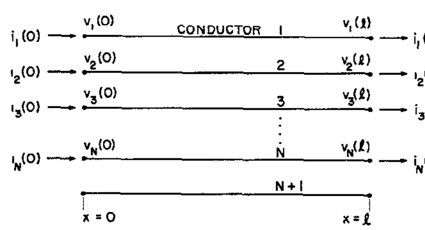

Fig. 1. $N+1$ coupled transmission lines.

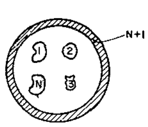

Fig. 2. Cross-sectional view of multiline

We now introduce incident and reflected wave amplitudes $(\boldsymbol{a})_p$ and $(\boldsymbol{b})_p$ ($p = 1, 2, 3, \cdots, 2N$)

$$\boldsymbol{a} = \left[\frac{\boldsymbol{a}_1}{\boldsymbol{a}_2}\right], \quad \boldsymbol{b} = \left[\frac{\boldsymbol{b}_1}{\boldsymbol{b}_2}\right]$$

with

$$(\boldsymbol{a}_1)_k = \frac{1}{2\sqrt{r_{k_0}}}(v_k(0) + r_{k_0}i_k(0)), \tag{8a}$$

$$(\boldsymbol{b}_1)_k = \frac{1}{2\sqrt{r_{k_0}}}(v_k(0) - r_{k_0}i_k(0)), \tag{8b}$$

$$(\boldsymbol{a}_2)_k = \frac{1}{2\sqrt{r_{k_l}}}(v_k(l) - r_{k_l}i_k(l)), \tag{8c}$$

$$(\boldsymbol{b}_2)_k = \frac{1}{2\sqrt{r_{k_l}}}(v_k(l) + r_{k_l}i_k(l)). \tag{8d}$$

The parameters r_{k_0} and r_{k_l} represent prescribed terminating resistances at their respective ports. The goal of this work is to choose the r's so that directional coupler operation is achieved.

Define a $2N$ by $2N$ transfer scattering matrix T_s

$$\left[\frac{\boldsymbol{b}_1}{\boldsymbol{a}_1}\right] = T_s\left[\frac{\boldsymbol{a}_2}{\boldsymbol{b}_2}\right]. \tag{9}$$

$$T_s = \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right], \tag{10}$$